

Plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable

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Abstract

Listed as No. 53 among the one hundred famous unsolved problems in [J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, Berlin, 2008] is Steinberg's conjecture, which states that every planar graph without 4- and 5-cycles is 3-colorable. In this paper, we show that plane graphs without 4- and 5-cycles are 3-colorable if they have no ext-triangular 7-cycles. This implies that (1) planar graphs without 4-, 5-, 7-cycles are 3-colorable, and (2) planar graphs without 4-, 5-, 8-cycles are 3-colorable, which cover a number of known results in the literature motivated by Steinberg's conjecture.

1 Introduction

In the field of 3-colorings of planar graphs, one of the most active topics is about a conjecture proposed by Steinberg in 1976: every planar graph without cycles of length 4 and 5 is 3-colorable. There had been no progress on this conjecture for a long time, until Erdős [14] suggested a relaxation of it: does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] confirmed that such k exists and $k \leq 11$. This result was later on improved to $k \leq 9$ by Borodin [2] and, independently, by Sanders and Zhao [13], and to $k \leq 7$ by Borodin, Glebov, Raspaud and Salavatipour [3].

Theorem 1.1 ([3]). *Planar graphs without cycles of length from 4 to 7 are 3-colorable.*

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We remark that Steinberg's conjecture was recently shown to be false in [6], by constructing a counterexample to the conjecture. The question whether every planar graph without cycles of length from 3 to 5 is 3-colorable is still open.

A more general problem than Steinberg's Conjecture was formulated in [11, 9]:

Problem 1.2. *What is \mathcal{A} , a set of integers between 5 and 9, such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor i is 3-colorable?*

Thus, Steinberg's Conjecture states that $5 \in \mathcal{A}$. Since so far no element of \mathcal{A} has been confirmed, it seems reasonable to consider a relaxation of Problem 1.2 where more integers are forbidden to be the length of a cycle in planar graphs. Due to a famous theorem of Grötzsch that planar graphs without triangles are 3-colorable, triangles are always allowed in further sufficient conditions. Several papers together contribute to the result below:

Theorem 1.3. *For any three integers i, j, k with $5 \leq i < j < k \leq 9$, it holds true that planar graphs having no cycles of length 4, i, j, k are 3-colorable.*

Later on, the sufficient conditions, concerning three integers forbidden to be the length of a cycle, were considered. The corresponding problem can be formulated as follows:

Problem 1.4. *What is \mathcal{B} , a set of pairs of integers (i, j) with $5 \leq i < j \leq 9$, such that planar graphs without cycles of length 4, i, j are 3-colorable?*

It has been proved by Borodin et al. [4] and independently by Xu [17] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable. Hence, $(5, 7) \in \mathcal{B}$, which improves on Theorem 1.1. More elements of \mathcal{B} have been confirmed: $(6, 8) \in \mathcal{B}$ by Wang and Chen [15], $(7, 9) \in \mathcal{B}$ by Lu et al. [11], and $(6, 9) \in \mathcal{B}$ by Jin et al. [9]. The result $(6, 7) \in \mathcal{B}$ is implied in the following theorem, which reconfirms the results $(5, 7) \in \mathcal{B}$ and $(6, 8) \in \mathcal{B}$.

Theorem 1.5 ([5]). *Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.*

In this paper, we show that $(5, 8) \in \mathcal{B}$, which leaves four pairs of integers $(5, 6), (5, 9), (7, 8), (8, 9)$ unconfirmed as elements of \mathcal{B} .

Recently, Mondal gave a proof of the result $(5, 8) \in \mathcal{B}$ in [12]. Here we exhibit two counterexamples to the theorem proved in that paper which yields the result $(5, 8) \in \mathcal{B}$. We restated this theorem as follows. Let C be a cycle of length at most 12 in a plane graph without 4-, 5- and 8-cycles. C is bad if it is of length 9 or 12 and the subgraph inside C has a partition into 3- and 6-cycles; otherwise, C is good.

Theorem 1.6 (Theorem 2 in [12]). *Let G be a graph without 4-, 5-, and 8-cycles. If D is a good cycle of G , then every proper 3-coloring of D can be extended to a proper 3-coloring of the whole graph G .*

Counterexamples to Theorem 1.6. A plane graph G_1 consisting of a cycle C of length 12, say $C := [v_1 \dots v_{12}]$, and a vertex u inside C connected to all of v_1, v_2, v_6 . The graph G_1 contradicts Theorem 1.6, since any proper 3-coloring of C where v_1, v_2, v_6 receive pairwise distinct colors can not be extended to G_1 . Also, a plane graph G_2 consisting of a cycle C of length 12 and a triangle T inside C , say $C := [v_1 \dots v_{12}]$ and $T := [u_1 u_2 u_3]$, and three more edges $u_1 v_1, u_2 v_4, u_3 v_7$. The graph G_2 contradicts Theorem 1.6, since any proper 3-coloring of C where v_1, v_4, v_7 receive the same color can not be extended to G_2 (see Figure 1).

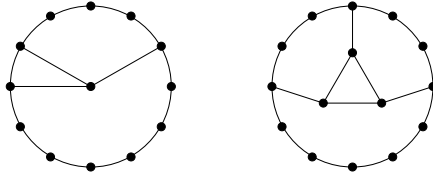


Figure 1: two graphs as counterexamples to Theorem 1.6.

1.1 Notations and formulation of the main theorem

The graphs considered in this paper are finite and simple. A graph is planar if it is embeddable into the Euclidean plane. A plane graph (G, Σ) is a planar graph G together with an embedding Σ of G into the Euclidean plane, that is, (G, Σ) is a particular drawing of G in the Euclidean plane. In what follows, we will always say a plane graph G instead of (G, Σ) , which causes no confusion since no two embeddings of the same graph G will be involved in.

Let G be a plane graph and C be a cycle of G . By $Int(C)$ (or $Ext(C)$) we denote the subgraph of G induced by the vertices lying inside (or outside) C . The cycle C is *separating* if neither $Int(C)$ nor $Ext(C)$ is empty. By $\overline{Int}(C)$ (or $\overline{Ext}(C)$) we denote the subgraph of G consisting of C and its interior (or exterior). The cycle C is *triangular* if it is adjacent to a triangle, and C is *ext-triangular* if it is adjacent to a triangle of $\overline{Ext}(C)$.

The following theorem is the main result of this paper.

Theorem 1.7. *Plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles are 3-colorable.*

As a consequence of Theorem 1.7, the following corollary holds true.

Corollary 1.8. *Planar graphs without cycles of length 4, 5, 8 are 3-colorable, that is, $(5, 8) \in \mathcal{B}$.*

We remark that Theorem 1.7 implies the known result that $(5, 7) \in \mathcal{B}$ as well.

Denote by $d(v)$ the degree of a vertex v , by $|P|$ the number of edges of a path P , by $|C|$ the length of a cycle C and by $d(f)$ the size of a face f . A k -vertex (or k^+ -vertex, or k^- -vertex) is a vertex v with $d(v) = k$ (or $d(v) \geq k$, or $d(v) \leq k$). Similar notations are used for paths, cycles, faces with $|P|, |C|, d(f)$ instead of $d(v)$, respectively.

Let $G[S]$ denote the subgraph of G induced by S with either $S \subseteq V(G)$ or $S \subseteq E(G)$. A *chord* of C is an edge of $\overline{Int}(C)$ that connects two nonconsecutive vertices on C . If $Int(C)$ has a vertex v with three neighbors v_1, v_2, v_3 on C , then $G[\{vv_1, vv_2, vv_3\}]$ is called a *claw* of C . If $Int(C)$ has two adjacent vertices u and v such that u has two neighbors u_1, u_2 on C and v has two neighbors v_1, v_2 on C , then $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$ is called a *biclaw* of C . If $Int(C)$ has three pairwise adjacent vertices u, v, w which has a neighbor u', v', w' on C respectively, then $G[\{uv, vw, uw, uu', vv', ww'\}]$ is called a *triclawn* of C . If G has four vertices x, u, v, w inside C and four vertices x_1, x_2, v_1, w_1 on C such that $S = \{uv, vw, wu, ux, xx_1, xx_2, vv_1, ww_1\} \subseteq E(G)$, then $G[S]$ is called a *combcrown* of C (see Figure 2).

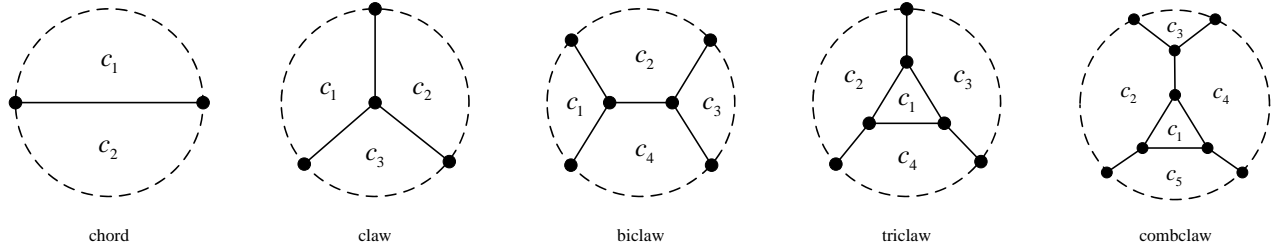


Figure 2: chord, claw, biclaw, triclawn and combcrown of a cycle

A *good cycle* is an 11^- -cycle that has none of claws, biclaws, triclaws and combclaws. A *bad cycle* is an 11^- -cycle that is not good.

Instead of Theorem 1.7, it is easier for us to prove the following stronger one:

Theorem 1.9. *Let G be a connected plane graph with neither 4- and 5-cycles nor ext-triangular 7-cycles. If D , the boundary of the exterior face of G , is a good cycle, then every proper 3-coloring of $G[V(D)]$ can be extended to a proper 3-coloring of G .*

The proof of Theorem 1.9 will be proceeded by using discharging method and is given in the next section. For more information on the discharging method, we refer readers to [7]. The rest of this section contributes to other needed notations.

Let C be a cycle and T be one of chords, claws, biclaws, triclaws and combclaws of C . We call the graph H consisting of C and T a *bad partition* of C . The boundary of any one of the parts, into which C is divided by H , is called a *cell* of H . Clearly, every cell is a cycle. In case of confusion, let us always order the cells c_1, \dots, c_t of H in the way as shown in Figure 2. Let k_i be the length of c_i . Then T is further called a (k_1, k_2) -chord, a (k_1, k_2, k_3) -claw, a (k_1, k_2, k_3, k_4) -biclaw, a (k_1, k_2, k_3, k_4) -triclaw and a $(k_1, k_2, k_3, k_4, k_5)$ -combclaw, respectively.

A vertex is *external* if it lies on the exterior face; *internal* otherwise. A vertex (or an edge) is *triangular* if it is incident with a triangle. We say a vertex is *bad* if it is an internal triangular 3-vertex; *good* otherwise. A path is a *splitting path* of a cycle C if it has the two end-vertices on C and all other vertices inside C . A k -cycle with vertices v_1, \dots, v_k in cyclic order is denoted by $[v_1 \dots v_k]$.

Let uvw be a path on the boundary of a face f of G with v internal. The vertex v is f -heavy if both uv and vw are triangular and $d(v) \geq 5$, and is f -Mlight if both uv and vw are triangular and $d(v) = 4$, and f -Vlight if neither uv nor vw is triangular and v is triangular and of degree 4. A vertex is f -light if it is either f -Mlight or f -Vlight.

Denote by \mathcal{G} the class of connected plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles.

2 The proof of Theorem 1.9

Suppose to the contrary that Theorem 1.9 is false. From now on, let G be a counterexample to Theorem 1.9 with fewest vertices. Thus, we may assume that the boundary D of the exterior face of G is a good cycle, and there exists a proper 3-coloring ϕ of $G[V(D)]$ which cannot be extended to a proper 3-coloring of G . By the minimality of G , we deduce that D has no chord.

2.1 Structural properties of the minimal counterexample G

Lemma 2.1. *Every internal vertex of G has degree at least 3.*

Proof. Suppose to the contrary that G has an internal vertex v with $d(v) \leq 2$. We can extend ϕ to $G - v$ by the minimality of G , and then to G by coloring v different from its neighbors. \square

Lemma 2.2. *G is 2-connected and therefore, the boundary of each face of G is a cycle.*

Proof. Otherwise, we can assume that G has a pendant block B with cut vertex v such that $B - v$ does not intersect with D . We first extend ϕ to $G - (B - v)$, and then 3-color B so that the color assigned to v is unchanged. \square

Lemma 2.3. *G has no separating good cycle.*

Proof. Suppose to the contrary that G has a separating good cycle C . We extend ϕ to $G - \text{Int}(C)$. Furthermore, since C is a good cycle, the color of C can be extended to its interior. \square

By the definition of a bad cycle, one can easily conclude the lemma as follows.

Lemma 2.4. *If C is a bad cycle of a graph in \mathcal{G} , then C has length either 9 or 11. Furthermore, if $|C| = 9$, then C has a $(3,6,6)$ -claw or a $(3,6,6,6)$ -triclawn; if $|C| = 11$, then C has a $(3,6,8)$ -claw, or a $(3,6,6,6)$ - or $(6,3,6,6)$ -biclaw, or a $(3,6,6,8)$ -triclawn, or a $(3,6,6,6,6)$ -combcawn.*

Notice that all 3- and 6- and 8-cycles of G are facial, thus the following statement is a consequence of the previous lemma together with the fact that $G \in \mathcal{G}$.

Lemma 2.5. *G has neither bad cycle with a chord nor ext-triangular bad 9-cycle.*

Lemma 2.6. *Let P be a splitting path of D which divides D into two cycles D' and D'' . The following four statements hold true.*

- (1) *If $|P| = 2$, then there is a triangle between D' and D'' .*
- (2) *$|P| \neq 3$.*
- (3) *If $|P| = 4$, then there is a 6- or 7-cycle between D' and D'' .*
- (4) *If $|P| = 5$, then there is a 9^- -cycle between D' and D'' .*

Proof. Since D has length at most 11, we have $|D'| + |D''| = |D| + 2|P| \leq 11 + 2|P|$.

(1) Let $P = xyz$. Suppose to the contrary that $|D'|, |D''| \geq 6$. By Lemma 2.1, y has a neighbor other than x and z , say y' . It follows that y' is internal since otherwise D is a bad cycle with a claw. Without loss of generality, let y' lie inside D' . Now D' is a separating cycle. By Lemma 2.3, D' is not good, i.e., either D' is bad or $|D'| \geq 12$. Since every bad cycle has length either 9 or 11 by Lemma 2.4, we have $|D'| \geq 9$. Recall that $|D'| + |D''| \leq 15$, thus $|D'| = 9$ and $|D''| = 6$. Now D' has either a $(3,6,6)$ -claw or a $(3,6,6,6)$ -triclawn by Lemma 2.4, which implies that D has a biclaw or a combclawn respectively, a contradiction.

(2) Suppose to the contrary that $|P| = 3$. Let $P = wxyz$. Clearly $|D'|, |D''| \geq 6$. Let x' and y' be a neighbor of x and y not on P , respectively. If both x' and y' are external, then D has a biclaw. Hence, we may assume x' lies inside D' . By Lemmas 2.3 and 2.4 and the inequality $|D'| + |D''| \leq 17$, we deduce that D' is a bad cycle and D'' is a good 8^- -cycle. If y' is internal, then y' lies inside D' . It follows with the specific interior of a bad cycle that $x' = y'$

and D' has either a claw or a biclaw, which implies that D has either a triclave or a combclaw respectively, a contradiction. Hence, y' is external. Since every bad cycle as well as every 6^- - or 8 -cycle contains no chord by Lemma 2.5, we deduce that yy' is a $(3,6)$ -chord of D'' . It follows that D' is a bad and ext-triangular 9-cycle, contradicting Lemma 2.5.

(3) Let $P = vwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 8$. Since $|D'| + |D''| \leq 19$, we have $|D'|, |D''| \leq 11$. Since G has no 4- and 5-cycles, if G has an edge e connecting two nonconsecutive vertices on P , then the cycle formed by e and P has to be a triangle, yielding a splitting 3-path of D , contradicting the statement (2). Therefore, no pair of nonconsecutive vertices on P are adjacent.

Let w', x', y' be a neighbor of w, x, y not on P , respectively. The statement (2) implies that x' is internal. Let x' lie inside of D' . Thus D' is a bad 9- or 11-cycle. If D' is a bad 11-cycle, then D'' is a facial 8-cycle, and thus both w' and y' lie in $\overline{Int}(D')$, which is impossible by the interior of a bad cycle. Hence, D' is a bad 9-cycle. By the statement (1), if $w' \in V(D'')$, then G has the triangle $[vw'w']$, which makes D' ext-triangular, a contradiction. Hence, $w' \notin V(D'')$. Furthermore, as a bad cycle, D' has no chord by Lemma 2.5, thus w' is internal. If w' lies inside D' , then it gives the interior of D' no other choices but $w' = x'$ and D' has a $(3,6,6)$ -claw, in which case this claw contains a splitting 3-path of D , a contradiction. Hence, w' lies inside D'' . Similarly, we can deduce that y' lies inside D'' as well. Note that $|D''| \in \{8, 9, 10\}$, thus D'' is a bad 9-cycle but has to contain both w' and y' inside, which is impossible.

(4) Let $P = uvwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 10$. Since $|D'| + |D''| \leq 21$, we have $|D'|, |D''| \leq 11$. By similar argument as in the proof of the statement (3), one can conclude that G has no edge connecting two nonconsecutive vertices on P . Let v', w', x', y' be a neighbor of v, w, x, y not on P , respectively.

The statement (2) implies that both w' and x' are internal. Let w' lie inside D' . It follows that D' is a bad 11-cycle and D'' is a 10-cycle. Thus x' also lies inside D' and furthermore, $x' = w'$ and D' is a bad cycle with either a $(3,6,8)$ -claw or a $(3,6,6,6)$ -biclaw. It follows that $v', y' \in V(D'')$. By the statement (1), G has two triangles $[uvv']$ and $[yy'z]$, at least one of them is adjacent to a 7-cycle of $\overline{Int}(D')$, a contradiction. \square

Lemma 2.7. *Let G' be a connected plane graph obtained from G by deleting a set of internal vertices and identifying two other vertices so that at most one pair of edges are merged. If we*

(a) *identify no two vertices of D , and create no edge connecting two vertices of D , and*

(b) *create no 6^- -cycle and ext-triangular 7-cycle,*

then ϕ can be extended to G' .

Proof. The item (a) guarantees that D is unchanged and bounds G' , and ϕ is a proper 3-coloring of $G'[V(D)]$. By item (b), the graph G' is simple and $G' \in \mathcal{G}$. Hence, to extend ϕ to G' by the minimality of G , it remains to show that D is a good cycle of G' .

Suppose to the contrary that D has a bad partition H in G' . Clearly, H has a 6-cell C' such that the intersection between D and C' is a path $v_1 \dots v_k$ of length $k - 1$ with $k \in \{4, 5\}$. Since we create no 6-cycles, C' corresponds to a 6-cycle C of the original graph G . Recall that at most one pair of edges are merged during the process from G to G' , we deduce that the intersection between D and C is a path P of one of the forms $v_1 \dots v_k, v_1 \dots v_{k-1}, v_2 \dots v_k$. Thus, $|P| \in \{3, 4, 5\}$. If $|P| \in \{4, 5\}$, then C contains a splitting 3- or 2-path of D in G , yielding a contradiction by Lemma 2.6. Hence, $|P| = 3$ and so $k = 4$. By the choice of the 6-cell C' , we may assume that the bad partition H has either a (3,6,6,6)- or (3,6,6,8)-triclawn. Now H contains three splitting 3-paths of D , at least one of them does not contain the identified vertex of G' no matter where it is, yielding the existence of a splitting 3-path of D in G , contradicting Lemma 2.6. \square

Lemma 2.8. *G has no edge uv incident with a 6-face and a 3-face such that both u and v are internal 3-vertices and therefore, every bad cycle of G has either a (3,6,6)- or (3,6,8)-clawn or a (3,6,6,6)-biclawn.*

Proof. Suppose to the contrary that such an edge uv exists. Denote by $[uvwxyz]$ and $[uv]$ the 6-face and 3-face, respectively. Lemma 2.6 implies that not both of w and z are external vertices. Without loss of generality, we may assume that w is internal. Let G' be the graph obtained from G by deleting u and v , and identifying w with y so that wx and yx are merged. Clearly, G' is a plane graph on fewer vertices than G . We will show that both the items in Lemma 2.7 are satisfied.

Since w is internal, we identify no two vertices on D . If we create an edge connecting two vertices on D , then w has a neighbor w_1 not adjacent to y and both y and w_1 are external. But now, Lemma 2.6 implies that x is external and thus, $[ww_1x]$ is a triangle which makes the 7-cycle $[utvwxyz]$ ext-triangular. Hence, the item (a) holds.

Suppose we create a 6⁻-cycle or an ext-triangular 7-cycle C' . Thus G has a 7⁻-path P between w and y corresponding to C' . If $x \in V(P)$, then neither wx nor xy are on P since otherwise, C' already exists in G . Hence, the paths wxy and P form two cycles, both of them has length at least 6. It follows that $|P| \geq 10$, a contradiction. Hence, we may assume that $x \notin V(P)$. The paths P and wxy form a 9⁻-cycle, say C . By Lemma 2.1, we may let x_1 be a neighbor of x other than y and w . We have $x_1 \notin V(P)$, since otherwise P has length at least 8. Now C has to contain either u and v or x_1 inside, which implies that C is a bad

9-cycle. By Lemma 2.5, C is not ext-triangular. Thus C' is a 7-cycle that is not ext-triangular, contradicting the supposition. Hence, the item (b) holds.

By Lemma 2.7, the pre-coloring ϕ can be extended to G' . Since z and w receive different colors, we can properly color v and u , extending ϕ further to G . \square

We follow the notations of M -face and MM -face in [3], and define weak tetrads. An M -face is an 8-face f containing no external vertices with boundary $[v_1 \dots v_8]$ such that the vertices $v_1, v_2, v_3, v_5, v_6, v_7$ are of degree 3 and the edges $v_1v_2, v_3v_4, v_4v_5, v_6v_7$ are triangular. An MM -face is an 8-face f containing no external vertices with boundary $[v_1 \dots v_8]$ such that v_2 and v_7 are of degree 4 and other six vertices on f are of degree 3, and the edges $v_1v_2, v_2v_3, v_4v_5, v_6v_7, v_7v_8$ are triangular. A weak tetrad is a path $v_1 \dots v_5$ on the boundary of a face f such that both the edges v_1v_2 and v_3v_4 are triangular, all of v_1, v_2, v_3, v_4 are internal 3-vertices, and v_5 is either of degree 3 or f -light.

Lemma 2.9. *G has no weak tetrad and therefore, every face of G contains no five consecutive bad vertices.*

Proof. Suppose to the contrary that G has a weak tetrad T following the notation used in the definition. Denote by v_0 the neighbor of v_1 on f with $v_0 \neq v_2$. Denote by x the common neighbor of v_1 and v_2 , and y the common neighbor of v_3 and v_4 . If $x = v_0$, then v_1 is an internal 2-vertex, contradicting Lemma 2.1. Hence, $x \neq v_0$ and similarly, $x \neq v_3$. Since G has no 4- or 5-cycles, $x \notin \{v_4, v_5\}$. Concluding above, $x \notin v_0 \cup V(T)$. Similarly, $y \notin v_0 \cup V(T)$. Moreover, $x \neq y$ since otherwise $[v_1v_2v_3x]$ is a 4-cycle. We delete v_1, \dots, v_4 and identify v_0 with y , obtaining a plane graph G' on fewer vertices than G . We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a 6^- -cycle or an ext-triangular 7-cycle C' . Thus G has a 7^- -path P between v_0 and y corresponding to C' . If $x \in V(P)$, then the cycle formed by P and v_0v_1x has length at least 6 and the one formed by P and xv_2v_3y has length at least 8, which gives $|P| \geq 9$, a contradiction. Hence, $x \notin V(P)$. The paths P and $v_0v_1v_2v_3y$ form a 11^- -cycle, say C . Now C contains either x or v_4 inside. Thus, C is a bad cycle. By Lemma 2.8, C has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw. Note that both the two faces incident with v_2v_3 has length at least 8, thus C has a bad partition owning an 8-cell no matter which one of x and v_4 lies inside C . It follows that C has a (3,6,8)-claw. If x lies inside C , then the 6-cell is adjacent to the triangle $[xv_1v_2]$ with $d(v_1) = d(x) = 3$, contradicting Lemma 2.8. Hence, v_4 lies inside C . Note that v_4v_5 is incident with the 6-cell and the 8-cell, we deduce that v_5 is not f -light. By the assumption of T as a weak tetrad, we may assume that $d(v_5) = 3$. We delete

v_5 together with other vertices of T and repeat the argument above, yielding a contradiction. Therefore, the item (b) holds.

Suppose we identify two vertices on D or create an edge connecting two vertices on D . Thus there is a splitting 4- or 5-path Q of D containing the path $v_0v_1v_2v_3y$. By Lemma 2.6, Q together with D forms a 9^- -cycle which corresponds to a 5^- -cycle in G' . Since we create no 6^- -cycle, a contradiction follows. Hence, the item (a) holds.

By Lemma 2.7, the pre-coloring ϕ can be extended to G' . We first properly color v_5 (if needed), v_4, v_3 in turn. Since v_0 and v_3 receive different colors, we can properly color v_1 and v_2 , extending ϕ further to G . \square

Lemma 2.10. *G has no M -face.*

Proof. Suppose to the contrary that G has an M -face f following the notation used in the definition. For $(i, j) \in \{(1, 2), (3, 4), (4, 5), (6, 7)\}$, denote by t_{ij} the common neighbor of v_i and v_j . By similar argument as in the proof of previous lemma, we deduce that the vertices $t_{12}, t_{34}, t_{45}, t_{67}$ are pairwise distinct and not incident with f . We delete $v_1, v_2, v_3, v_5, v_6, v_7$ and identify v_4 with v_8 , obtaining a plane graph G' on fewer vertices than G . We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a 6^- -cycle or an ext-triangular 7-cycle C' . Thus G has a 7^- -path P between v_4 and v_8 corresponding to C' . By the symmetry of an M -face, we may assume that P together with the path $v_4 \dots v_8$ forms a 11^- -cycle C containing v_1, v_2, v_3 inside. It follows with Lemma 2.8 that C is a bad cycle with a $(3, 6, 6, 6)$ -claw. But now $\overline{Int}(C)$ contains f that is an 8-face, a contradiction. Therefore, the item (b) holds.

The satisfaction of the item (a) can be proved in a similar way as in the proof of previous lemma.

By Lemma 2.7, the pre-coloring ϕ can be extended to G' . Since we first color v_3 different from v_8 , both v_1 and v_2 can be properly colored. Finally, color v_5, v_6, v_7 in the same way, extending ϕ further to G . \square

Lemma 2.11. *G has no MM -face.*

Proof. Suppose to the contrary that G has an MM -face f following the notation used in the definition. For $(i, j) \in \{(1, 2), (2, 3), (4, 5), (6, 7), (7, 8)\}$, denote by t_{ij} the common neighbor of v_i and v_j . Similarly, we deduce that the vertices $t_{12}, t_{23}, t_{45}, t_{67}, t_{78}$ are pairwise distinct and not incident with f . We delete all the vertices of f and identify t_{12} with t_{67} , obtaining a plane graph G' on fewer vertices than G . To extend ϕ to G' , it suffices to fulfill the item (a) of Lemma 2.7, as what we did in previous lemma.

Suppose that we create a 6^- -cycle or an ext-triangular 7-cycle C' . Thus G has a 7^- -path P between t_{12} and t_{67} corresponding to C' . If $t_{78} \in V(P)$, then both the cycles formed by P and $t_{12}v_1v_8t_{78}$ and by P and $t_{78}v_7t_{67}$ have length at least 8, which gives $|P| \geq 11$, a contradiction. Hence, $t_{78} \notin V(P)$. The paths P and $t_{12}v_1v_8v_7t_{67}$ form a 11^- -cycle, say C . It follows that C is a bad cycle containing either t_{78} or v_2, \dots, v_6 inside, that is, either C has a bad partition owning two 8^+ -cell or C contains five vertices inside, a contradiction in any case.

We further extend ϕ from G' to G as follows. Let α, β and γ be the three colors used in ϕ . First regardless the edge v_1v_8 , we can properly color v_2, v_1, v_3 and v_7, v_8, v_6 . If v_1 and v_8 receive different colors and so do v_3 and v_6 , then v_4 and v_5 can be properly colored, we are done. Hence, we may assume without loss of generality that v_1 and v_8 receive the same color, say β . Let α be the color assigned to t_{12} and t_{67} . Thus v_2 and v_7 are colored with γ and t_{78} is colored with α . We recolor v_8, v_7, v_6 with γ, β, γ respectively. Now v_1 and v_8 receive different colors and so do v_3 and v_6 . Again v_4 and v_5 can be properly colored, we are also done. \square

2.2 Discharging in G

Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G . Denote by f_0 the exterior face of G . Give initial charge $ch(x)$ to each element x of $V \cup F$, where $ch(f_0) = d(f_0) + 4$, $ch(v) = d(v) - 4$ for $v \in V$, and $ch(f) = d(f) - 4$ for $f \in F \setminus \{f_0\}$. Discharge the elements of $V \cup F$ according to the following rules:

- R1. Every internal 3-face receives $\frac{1}{3}$ from each incident vertex.
- R2. Every internal 6^+ -face sends $\frac{2}{3}$ to each incident 2-vertex.
- R3. Every internal 6^+ -face sends each incident 3-vertex v charge $\frac{2}{3}$ if v is triangular, and charge $\frac{1}{3}$ otherwise.
- R4. Every internal 6^+ -face f sends $\frac{1}{3}$ to each f -light vertex, and receives $\frac{1}{3}$ from each f -heavy vertex.
- R5. Every internal 6^+ -face receives $\frac{1}{3}$ from each incident external 4^+ -vertex.
- R6. The exterior face f_0 sends $\frac{4}{3}$ to each incident vertex.

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ after discharging. On one hand, by Euler's formula we deduce $\sum_{x \in V \cup F} ch(x) = 0$. Since the sum of charges over all elements of $V \cup F$ is unchanged, it follows that $\sum_{x \in V \cup F} ch^*(x) = 0$. On the other hand, we show that

$ch^*(x) \geq 0$ for $x \in V \cup F \setminus \{f_0\}$ and $ch^*(f_0) > 0$. Hence, this obvious contradiction completes the proof of Theorem 1.9. It remains to show that $ch^*(x) \geq 0$ for $x \in V \cup F \setminus \{f_0\}$ and $ch^*(f_0) > 0$.

We remark that the discharging rules can be tracked back to the one used in [3].

Lemma 2.12. $ch^*(v) \geq 0$ for $v \in V$.

Proof. First suppose that v is external. Since D is a cycle, $d(v) \geq 2$. If $d(v) = 2$, then since D has no chord, the internal face incident with v is not a triangle and sends $\frac{2}{3}$ to v by $R2$. Moreover, v receives $\frac{4}{3}$ from f_0 by $R6$, which gives $ch^*(v) = d(v) - 4 + \frac{2}{3} + \frac{4}{3} = 0$. If $d(v) = 3$, then v sends charge to at most one 3-face by $R1$ and thus $ch^*(v) \geq d(v) - 4 - \frac{1}{3} + \frac{4}{3} = 0$. If $d(v) \geq 4$, then v sends at most $\frac{1}{3}$ to each incident internal face by $R1$ and $R5$, yielding $ch^*(v) \geq d(v) - 4 - \frac{1}{3}(d(v) - 1) + \frac{4}{3} > 0$. Hence, we are done in any case.

It remains to suppose that v is internal. By Lemma 2.1, $d(v) \geq 3$. If $d(v) = 3$, then we have $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{2}{3} \times 2 = 0$ by $R1$ and $R3$ when v is triangular, and $ch^*(v) = d(v) - 4 + \frac{1}{3} \times 3 = 0$ by $R3$ when v not. If $d(v) = 4$, then v is incident with k 3-faces with $k \leq 2$. By $R1$ and $R4$, we have $ch^*(v) = d(v) - 4 - \frac{1}{3} \times 2 + \frac{1}{3} \times 2 = 0$ when $k = 2$, $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{1}{3} = 0$ when $k = 1$, and $ch^*(v) = d(v) - 4 = 0$ when $k = 0$. If $d(v) = 5$, then v sends charge to at most two 3-faces by $R1$ and to at most one 6^+ -face by $R4$, which gives $ch^*(v) \geq d(v) - 4 - \frac{1}{3} \times 2 - \frac{1}{3} = 0$. Hence, we may next assume that $d(v) \geq 6$. Since v sends at most $\frac{1}{3}$ to each incident face by our rules, we get $ch^*(v) \geq d(v) - 4 - \frac{1}{3}d(v) \geq 0$. \square

Lemma 2.13. $ch^*(f_0) > 0$.

Proof. Recall that $ch(f_0) = d(f_0) + 4$ and $d(f_0) \leq 11$. We have $ch^*(f_0) \geq d(f_0) + 4 - \frac{4}{3}d(f_0) > 0$ by $R6$. \square

Lemma 2.14. $ch^*(f) \geq 0$ for $f \in F \setminus \{f_0\}$.

Proof. We distinguish cases according to the size of f . Since G has no 4- and 5-cycle, $d(f) \notin \{4, 5\}$.

If $d(f) = 3$, then f receives $\frac{1}{3}$ from each incident vertices by $R1$, which gives $ch^*(f) = d(f) - 4 + \frac{1}{3} \times 3 = 0$.

Let $d(f) = 6$. For any incident vertex v , by the rules, f sends to v charge $\frac{2}{3}$ if v is either of degree 2 or bad, and charge at most $\frac{1}{3}$ otherwise. Since G has no ext-triangular 7-cycles, f is adjacent to at most one 3-face. Furthermore, by Lemma 2.8, f contains at most one bad vertex. If f contains a 2-vertex, say u , we can deduce with Lemma 2.6 that u is the unique 2-vertex of f and the two neighbors of u on f are external 3^+ -vertices which receive nothing

from f . It follows that $ch^*(f) \geq d(f) - 4 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \times 2 = 0$. Hence, we may assume that f contains no 2-vertices. If f has no bad vertices, then f sends each incident vertex at most $\frac{1}{3}$, which gives $ch^*(f) \geq d(f) - 4 - \frac{1}{3}d(f) = 0$. Hence, we may let x be a bad vertex of f . Denote by y the other common vertex between f and the triangle adjacent to f . By Lemma 2.8 again, y is not a bad vertex, i.e., y is either an internal 4^+ -vertex or an external 3^+ -vertex. By our rules, f sends nothing to y , yielding $ch^*(f) \geq d(f) - 4 - \frac{2}{3} - \frac{1}{3} \times 4 = 0$.

Let $d(f) = 7$. Since G has no ext-triangular 7-cycles, f contains no bad vertices. Moreover, by Lemma 2.6, we deduce that f has at most two 2-vertices. Thus, $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 2 - \frac{1}{3} \times 5 = 0$.

Let $d(f) \geq 8$. On the hand, if f contains precisely one external vertex, say w , then $d(w) \geq 4$ and so f receives $\frac{1}{3}$ from w by R5. Furthermore, since f contains no weak tetrad by Lemma 2.9, f has a good vertex other than w and sends at most $\frac{1}{3}$ to it. Hence, $ch^*(f) \geq d(f) - 4 + \frac{1}{3} - \frac{1}{3} - \frac{2}{3}(d(f) - 2) \geq 0$. On the other hand, if f contains at least two external vertices, then at least two of them are of degree more than 2. Since f sends nothing to external 3^+ -vertices, we have $ch^*(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) \geq 0$. By the two hands above, we may assume that all the vertices of f are internal. We distinguish two cases.

Case 1: assume that $d(f) = 8$. Denote by r the number of bad vertices of f . We have $ch^*(f) \geq d(f) - 4 - \frac{2}{3}r - \frac{1}{3}(d(f) - r) = \frac{4-r}{3} \geq 0$, provided by $r \leq 4$. Since f contains no weak tetrad, $r \leq 6$. Hence, we may assume that $r \in \{5, 6\}$. For $r = 5$, we claim that f has a vertex failing to take charge from f , which gives $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 5 - \frac{1}{3} \times 2 = 0$. Suppose to the contrary that no such vertex exists. Thus, the bad vertices of f can be paired so that any good vertex of the path of f between each pair is f -Mlight, contradicting the parity of r . For $r = 6$, since again f contains no five consecutive bad vertices, these six bad vertices of f are divided by the two good ones into cyclically either 3+3 or 2+4. We may assume that f has a good vertex that is either f -light or of degree 3, since otherwise we are done with $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 6 = 0$. Denote by u such a good vertex and by v the other one. By the drawing of u and of the 3-faces adjacent to f , we deduce that, for the case 3+3, f is an M -face, contradicting Lemma 2.10, and for the case 2+4, if u is f -Mlight then either f is an MM -face or v is f -heavy; otherwise f contains a weak tetrad. It follows with Lemmas 2.11 and 2.9 that v is f -heavy, which is the only possible case. Hence, f receives $\frac{1}{3}$ from v by R4, yielding $ch^*(f) \geq ch(f) - 4 - \frac{2}{3} \times 6 + \frac{1}{3} - \frac{1}{3} = 0$.

Case 2: assume that $d(f) \geq 9$. By Lemma 2.9, we deduce that f contains at least two good vertices, each of them receives at most $\frac{1}{3}$ from f . Thus, $ch^*(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) - \frac{1}{3} \times 2 = \frac{d(f)-10}{3} \geq 0$, provided by $d(f) \geq 10$. It remains to suppose $d(f) = 9$. If f

has at most six bad vertices, then $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$. Hence, we may assume that f has precisely seven bad vertices. By the same argument as for the case $d(f) = 8$ and f has five bad vertices above, f has a vertex failing to take charge from f , which gives $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times 7 - \frac{1}{3} = 0$. \square

By the previous three lemmas, the proof of Theorem 1.9 is completed.

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